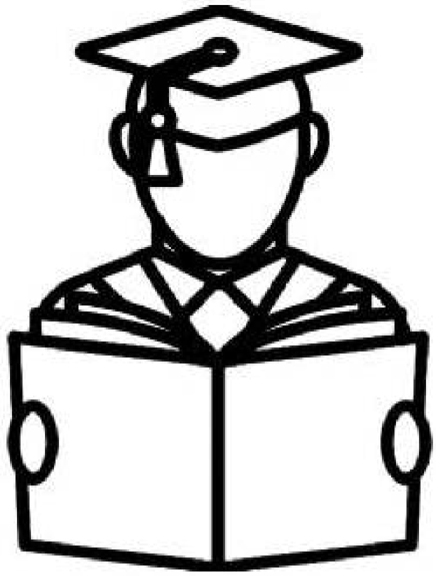

Mathematical Science for CSIR NET

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Developing Excellence ...

ANAND INSTITUTE OF MATHEMATICS

Founder & Director
Anand Kumar
(M.Tech., Ph.D.(D), JNU)

HIMANSHOO

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Summary → This notes was written under guidance of Anand Sir. I tried my best to keep it error free, but in case any mistake is found then no one will responsible for it, as it may just Human Error.

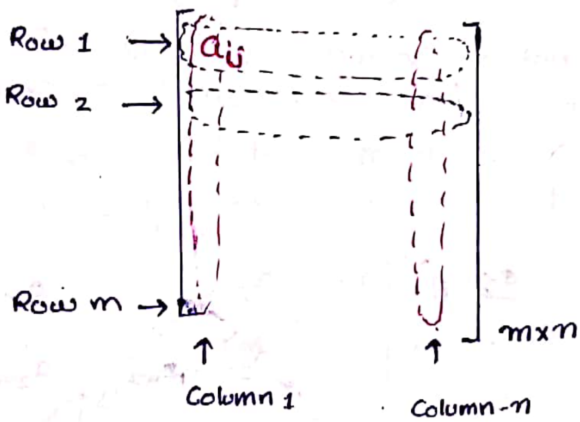
πAIM

1st lec
23/09

* Matrices →

"A set of mn numbers arranged in form of rectangular array consisting of m rows and n-columns is known as $m \times n$ matrix or matrix of order $m \times n$."

NOTE→ ① Usually matrices are denoted by capital letters of alphabet in bold type. and the element (numbers) consisting are closed within [ ] or ( )



a_{ij} → column no.
→ row no.

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⑪ In short: $A = [a_{ij}]_{m\times n} = (a_{ij})_{m \times n} = \| a_{ij} \|_{m \times n}$

$1 \leq i \leq m$
$1 \leq j \leq n$

e.g. $A = [ij]_{2\times 3}$

$$= \begin{bmatrix} 1.1 & 1.2 & 1.3 \\ 2.1 & 2.2 & 2.3 \end{bmatrix}_{2\times 3}$$

* Matrix over a Field →

$A = (a_{ij})_{m \times n}$ is s.t.b matrix over a field F if $a_{ij} \in F \ \forall \ i, j$. and matrix is known as F-matrix.

e.g. If $F = \mathbb{C}$ then A is known as complex matrix

If $F = \mathbb{R}$ then A is known as Real matrix.

NOTE→ By default if field of a matrix is not mentioned then it is complex field.

Some Special Type of matrices →

Row matrix → A matrix having exactly one row but any no. of columns

e.g. $[1 \quad 3 \quad 5 \quad 7]_{1\times 4}$

Column Matrix → A matrix having exactly one column but any no of rows.

e.g. $\begin{bmatrix} -1 \\ -5 \\ -7i \end{bmatrix}_{3\times 1}$

Null Matrix → A matrix whose each element or entry is equal to zero is known as Null matrix or Zero matrix.

It is denoted by $O = O_{m\times n} = [0]_{m\times n}$

e.g. $\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}_{2\times 3}$

Square matrix → A matrix having equal no. of row(s) and column(s) is known as square matrix i.e. $A = [a_{ij}]_{m\times n}$ is square matrix if $m = n$. And it is said to be square matrix of order n or n-rowed square matrix or n-columned sq. matrix.

e.g. $A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & & & \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}_{n\times n}$ is sq. matrix of order n.

$j > i$ → Super diagonal

$i = j$ → Principal Diagonal

$j < i$ → Sub Diagonal

Then:

Principal Diagonal Line → The line along which elts of the form a_{ii} (or $a_{ij} \mid i=j$) lies is known as Principal Diagonal lines. of $A = [a_{ij}]$, $i = 1, 2, \ldots n$

And the elements along P.D.L is known as diagonal elts of A.

Note → If order$(A) = n$ the number of elements in A is equal to $\underline{\underline{n}}$.

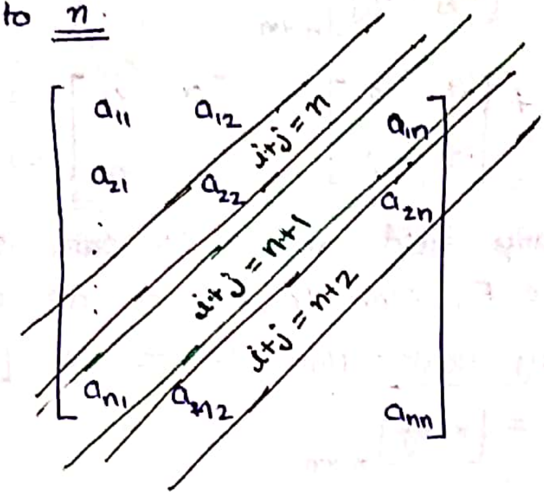
Super Diagonal Line → The line along which elements of form $a_{i\,i+1}$ or $(a_{ij} \mid i = j+1)$ lies is known as Super Diagonal lines of $A = [a_{ij}]_{n\times n}$

Note → If order $(A) = n$, the number of super diagonal elts of $A = \underline{\underline{n-1}}$, $i = 1, 2, \ldots n-1$

Sub Diagonal Line → The line along which elts of the form $a_{i\,i-1}$ (or $a_{ij} \mid j = i-1$).

Non-Principal Diagonal Line → The line along which elements of the form $(a_{ij} \mid i+j = n+1)$ lies is known as Non-Principal Diagonal Line of $A = [a_{ij}]_{n\times n}$

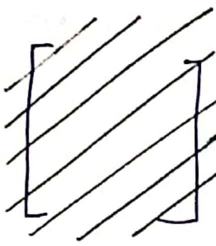
NOTE → If $o(A) = n$ then numbers of elements of A along N.P.D.L is equal to n.



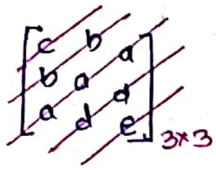
Q. Maximum possible number of distinct entries in $A = [a_{ij}]_{n\times n}$ in which $a_{pq} = a_{rs}$, whenever $p+q = r+s$ is equal to:

1. n^2 2. n ✓3. $2n-1$ 4. None

Sol^n →



→ No. of distinct lines we can make.



5 i.e $3\times 3-1$

$\because$ $2 \le \text{sum} \le 2n$

$\therefore$ No. of lines $2n-1$

* Algebra of Matrices →

1. **Equality of matrices →** Two matrices are conformable for being equal if they are of same order and further if $A = [a_{ij}]_{m\times n}$ & $B = [b_{ij}]_{m\times n}$ then $A = B$ iff $a_{ij} = b_{ij}$ $\forall\, i, j$

Q. For what values of a and b matrices

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \quad \& \quad B = \begin{bmatrix} 6 & 12 \\ a & b \end{bmatrix}$$

In Z_5: $A = B$ if $a = 3+5k$, $b = 4+5t$

Sol^n: $A = B$ iff $\underbrace{1=6}_{\times}$, $\underbrace{2=12}_{\times}$, $3=a$, $4=b$ $\therefore$ No. values of a & b.

2. **Sum or Add^n of two matrices →** Two matrices are conformable for add^n if they are of same size and if $A = [a_{ij}]_{m\times n}$ & $B = [b_{ij}]_{m\times n}$

Then, $A+B = C = [c_{ij}]_{m\times n}$ s.t. $c_{ij} = a_{ij} + b_{ij} \quad \forall\, i, j$

i.e. $A+B = [a_{ij}+b_{ij}]_{m\times n}$

e.g. $\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} + \begin{bmatrix} -1 & -2 \\ -3 & -4 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

NOTE → If F is any field and G is ~~said~~ set of all matrices of order $m\times n$ over F, then $(G,+)$ is an abelian Group.

3. **Multiplication of matrix by scalar Number →** If $A = [a_{ij}]_{m\times n}$ and k is any number then $kA = [k a_{ij}]_{m\times n}$

e.g. if $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ Then $(1+i)A = \begin{bmatrix} 1+i & 2+2i \\ 3+3i & 4+4i \end{bmatrix}$

NOTE → If k is any no. and $A = [a_{ij}]_{m\times n}$ & $B = [b_{ij}]_{m\times n}$

Then $\boxed{k(A+B) = kA + kB}$

Pf → $k(A+B) = [k(a_{ij}+b_{ij})]_{n\times n}$

$kA + kB = [k a_{ij}] + [k b_{ij}]$

$= [k(a_{ij}+b_{ij})]_{n\times n}$

$\Big\}\; k(A+B) = kA + kB$

4. **Multiplication or Product of Two matrices →** Two matrices A and B are conformable for multiplication in order if number of columns in A is equal to no. of rows in B i.e. AB exist if $A = [a_{ij}]_{m\times n}$ & $B = [b_{jk}]_{n\times p}$

and $AB = C = [c_{ik}]_{m\times p}$ where $c_{ik} = \sum_{j=1}^{n} a_{ij} b_{jk}$

Q. Let $A = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$. Then find AB.

Sol^n → $AB = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

$BA = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

NOTE→ (i) $AB=0$ need not imply that either A or B is null matrix.

(ii) In General $AB \neq BA$

(iii) Two matrices are said to be commute with each other if $AB=BA$

(iv) Two matrices are said to be anti-commute with each other if $AB=-BA$

(v) Multiplication of matrices is associative if conformability of matrices for multiplication is assured.

i.e. if $A=[a_{ij}]_{m\times n}$, $B=[b_{jk}]_{n\times p}$, $C=[c_{kl}]_{p\times q}$

Then $(AB)C = A(BC)$

Pf→ $(i,k)^{th}$ elt of $AB = \sum_{j=1}^{n} a_{ij} b_{jk}$

$(i,l)^{th}$ elt of $(AB)C = \sum_{k=1}^{p}\left(\sum_{j=1}^{n} a_{ij} b_{jk}\right) c_{kl}$ ....(i)

$(j,l)^{th}$ elt of $(BC) = \sum_{k=1}^{p} b_{jk} c_{kl}$

$(i,l)^{th}$ elt of $A(BC) = \sum_{j=1}^{n} a_{ij}\left(\sum_{k=1}^{p} b_{jk} c_{kl}\right)$ ...(ii)

From (i), Changing the order of summation:

$(i,l)^{th}$ elt of $(AB)C = \sum_{k=1}^{p}\sum_{j=1}^{n} a_{ij} b_{jk} c_{kl}$

$= \sum_{j=1}^{n} a_{ij}\left(\sum_{k=1}^{p} b_{jk} c_{kl}\right)$

From (i) and (ii), $\underline{(AB)C = A(BC)}$

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5. **Positive Integral Power of a Square Matrix→** If $A=(a_{ij})_{n\times n}$

Then $A_{n\times n}\cdot A_{n\times n}$ exist and $A\cdot A = A^2$. & $A^k = \underbrace{A\cdot A\cdot \,\cdots\, A}_{k\text{-times.}}$

NOTE→ (i) If A and B are two matrices then,

$(A+B)^2 = A^2 + AB + BA + B^2$ (using Bino)

(ii) If A and B are matrices of same order which commute with each other and $n\in \mathbb{N}$ then,

$$(A+B)^n = A^n + {}^nC_1 A^{n-1} B + {}^nC_2 A^{n-2} B^2 + \cdots + B^n$$

(iii) If $A = [a_{ij}]_{m\times n}$ whose each row sum to a and $B = [b_{jk}]_{n\times p}$ whose each row sum sum to b

Then each row sum of AB will be ab.

Pf→ $(i,k)^{th}$ elt of $AB = \sum_{j=1}^{n} a_{ij} b_{jk}$

Sum of elt of in ith row of $AB = \sum_{k=1}^{p}\left(\sum_{j=1}^{n} a_{ij} b_{jk}\right)$

Now, interchanging the order summation we get,

Sum of elt of in i-th row of $AB = \sum_{j=1}^{n} a_{ij}\left(\sum_{k=1}^{p} b_{jk}\right)$

$$= b\sum_{j=1}^{n} a_{ij}$$

$$= ba = \underline{\underline{ab}}$$

(iv) If $A = [a_{ij}]_{m\times n}$ whose each ~~row~~ column sum ~~to~~ a, $B = [b_{ij}]_{n\times p}$ whose each column sum to b, then each colum sum of AB will be ab.

Pf→ $(i,k)^{th}$ elt of $AB = \sum_{j=1}^{n} a_{ij} b_{jk}$

Sum of elts in k-th column $= \sum_{i=1}^{m}\left(\sum_{j=1}^{n} a_{ij} b_{jk}\right)$

Now, interchanging order of summation,

$$= \sum_{j=1}^{m} b_{jk}\left(\sum_{j=1}^{m} a_{ij}\right)$$

$$= \sum_{j=1}^{m} b_{jk}\,(a)$$

$$= ba = \underline{\underline{ab}}$$

(v) If sum of the elts of each row or column of a sq. matrix A is 'a' Then sum of each row/column element of A^n is a^n.

p.f. $\underbrace{\underbrace{\underbrace{A.A}_{a^2}.A}_{a^3} \cdots A_n}_{a^n}$

ABSTRACT ALGEBRA

FUNDAMENTAL SETS :-

$N = \{1, 2, 3, \ldots\}$ → set of natural number

and so on

and so on used for finite or infinite sets but upto infinity is used for infinite sets.

$Z = \{0, \pm 1, \pm 2, \pm 3 \ldots\}$ → set of integers.

$Q = \{\frac{m}{n} \mid m, n \in Z, n \neq 0\}$ → set of rational no.

R = set of real no. $-\infty$ —— $+\infty$

All numbers which lies on the number line.

$C = \{x + iy ; x, y \in R\}$ → set of Complex no.

Field starts from Q and after that R & C are extension of field.

N and Z are never field.

Q → smallest field

C → largest extension field.

(1) K is an extension field of R such that $R \subseteq K \subseteq C$ then K is 2.

(2) K is an extension field of Q such that $Q \subseteq K \subseteq R$ then K has infinite possibilities.

Reason :- $Q(\sqrt{2}) = \{a + b\sqrt{2} \mid a, b \in Q\}$

$Q(\sqrt{3}) = \{a + b\sqrt{3} \mid a, b \in Q\}$

$Q(\sqrt{p}) = \{a + b\sqrt{p} \mid a, b \in Q\}$ where p is any prime and primes are infinite

$\boxed{Q \subseteq Q(\sqrt{p}) \subseteq R}$

Dimension = deg. of extension = no. of free variables

$\mathbb{Q}(2)^{1/3} = \{a + b(2)^{1/3} + c(2)^{1/3} \mid a, b, c \in \mathbb{Q}\}$

$\mathbb{Q}(3)^{1/4} = \{a + b(3)^{1/4} + c(3)^{2/4} + d(3)^{3/4} \mid a, b, c, d \in \mathbb{Q}\}$

$\mathbb{Q}(1)^{1/4} = \{a + b(1)^{1/4} \mid a, b \in \mathbb{Q}\}$

$f(x) = x^4 - 1 = (x^2-1)(x^2+1)$
$= (x-1)(x+1)(x^2+1)$

This is the highest deg. polynomial

$= \{a + b(i) \mid a, b \in \mathbb{Q}\}$ $\left[\because (1)^{1/4} = \cos\frac{2\pi}{4} + i\sin\frac{2\pi}{4} = i\right]$

$= \mathbb{Q}[i] \rightarrow f(x) = x^2+1$

i.e. $\mathbb{Q}(1)^{1/4}$ is equivalent to $\mathbb{Q}[i]$

$R[i]$ polynomial is $f(x) = x^2+1$

x^2+1 cannot be factor over R then # of variable in $R[i]$ over $R = 2$.

notation of no.

$= \{a + ib \mid a, b \in R\}$

$= \mathbb{C}$

* If $\deg.[f(x)] = n$ then $\exists$ exactly n complex roots of $f(x) = 0$

let $a_1, a_2, \ldots a_n$ are n complex roots of $f(x) = 0$

then $f(x) = (x-a_1)(x-a_2) \cdots (x-a_n)$

then degree of polynomial or extension is 1.

$\mathbb{C}(a) = \{x \mid x \in \mathbb{C}\} = \mathbb{C}$

So we can ~~write~~ see extension of $\mathbb{C}$ is also $\mathbb{C}$

$10^5 - 9 = 10^5 - 9 - 1 + (1)$ remainder $= 1$

so we can say $\mathbb{C}$ is the highest extension.

<u>Ques.</u> Construct $\mathbb{C}(2)^{1/3}$ over $\mathbb{C}$

$\mathbb{C}(2)^{1/3} = f(x) = x^3 - 2 = (x-a_1)(x-a_2) \cdots (x-a_3)$ over $\mathbb{C}$.

$a_1, a_2, a_3 \in \mathbb{Q}$.

then # of free variable is 1.

$\mathbb{C}(2)^{1/3} = \{1.a \mid a \in \mathbb{C}\} = \mathbb{C}$
$= \mathbb{C}$

<u>Ques</u> Construct $\mathbb{C}[i]$ over $\mathbb{C}$.

$\mathbb{C}[i] \rightarrow$
$f(x) = x^2 + 1 = (x+i)(x-i)$ over $\mathbb{C}$.

then $\mathbb{C}[i]$ no. of variable $= 1$

$\Rightarrow \mathbb{C}[i] = \{1.a \mid a \in \mathbb{C}\}$
$= \mathbb{C}$

<u>Construction of Z_n</u>

$Z_n = \{0, 1, 2 \cdots n-1\} \rightarrow$ set of integers modulo

<u>OR</u> set of all residual classes modulo n.

Cardinality of Z_n is n because it contains $[0, 1, 2 \cdots n-1]$, Total $= n$

$Z_n = \{0, 1, 2, \cdots n-1\}$ or
$= \{\bar{0}, \bar{1}, \bar{2}, \cdots \overline{n-1}\}$ or

$= \{[0], [1], [2], [3] \ldots [n-1]\}$

Representation of $[0]$ in mathematical terms

$[0] \xrightarrow{\text{(class of 0)}}$

$[0] = 0 = \{x \in \mathbb{Z} \mid n / x - 0\}$

$[1] = 1 = \{x \in \mathbb{Z} \mid n / x - 1\}$

$[2] = 2 = \{x \in \mathbb{Z} \mid n / x - 2\}$

$\vdots$

$[n-1] = n-1 = \{x \in \mathbb{Z} \mid n / x - (n-1)\}$

then $\mathbb{Z}_n = \{0, 1, 2, \ldots n-1\}$

Cardinality of $\mathbb{Z}_n$

$\Rightarrow o(\mathbb{Z}_n) = |\mathbb{Z}_n| = n \; ; \; n \geq 1.$

order or Cardinality of $\mathbb{Z}_n$ is n.

Example:- $\mathbb{Z}_6 = \{0, 1, 2, 3, 4, 5\}$

then $o(\mathbb{Z}_6) = |\mathbb{Z}_6| = 6$

$[0] = \{x \in \mathbb{Z} \mid 6 / x - 0\} = \{-18, -12, -6, 0, 6, 12 \ldots\}$

$[1] = \{x \in \mathbb{Z} \mid 6 / x - 1\} = \{-17, -11, -5, 1, 7, 13, 19 \ldots\}$

$[2] = \{x \in \mathbb{Z} \mid 6 / x - 2\} = \{-16, -10, -4, 2, 8, 14, 20 \ldots\}$

$[3] = \{x \in \mathbb{Z} \mid 6 / x - 3\} = \{-15, -9, -3, 3, 9, 15, 21 \ldots\}$

$[4] = \{x \in \mathbb{Z} \mid 6 / x - 4\} = \{\ldots -14, -8, -2, 4, 10, 16, 22 \ldots\}$

$[5] = \{x \in \mathbb{Z} \mid 6 / x - 5\} = \{\ldots -13, -7, -1, 5, 11, 17, 23 \ldots\}$

then $Z_6 = \{0, 1, 2, 3, 4, 5\}$

Ques:- $R = Z_5$ then $\frac{1}{3} \in Z_5$?

Sol^n:- if $a|b$ then $\exists\ x \in R$ such that $b = ax$

if $3|1$ then $\exists\ x \in Z_5$ such that $1 = 3x$

$\Rightarrow 1 = 3 \cdot 2 \ , \ 2 \in Z_5$

$\Rightarrow \frac{1}{3} = 2 \in Z_5$

$\Rightarrow \frac{1}{3} \in Z_5$

Ques $R = Z_6$ then $\frac{1}{3} \in Z_6$?

Sol^n. if $a|b$ then $\exists\ x \in R$ s.t. $b = ax$.
if $3|1$ then $\exists\ x \in Z_6$ s.t. $1 = 3x$; $x \in Z_6$

But there is no $x \in Z_6$ s.t. $1 = 3x$
then $\frac{1}{3} \notin Z_6$.

Note:- if $a|b$ then $\exists\ x \in Z_n$ s.t. $b = ax$
if gcd of $(a,n) | b$ then $x \in Z_n$
otherwise $x \notin Z_n$

For example $\frac{1}{7} \in Z_{10}$?

gcd of $(7, 10) = 1$
and 1 divides 1
so $\frac{1}{7} \in Z_{10}$

i.e. $\frac{1}{7} = 3 \in Z_{10}$

$\Rightarrow \frac{1}{7} \in Z_{10}$

Ques $\frac{3}{8} \in Z_{10}$?

gcd of $(8, 10) = 2$ but $2 \nmid 3$
then $\frac{3}{8} \notin Z_{10}$

Ques $\frac{2}{8} \in Z_{10}$?

gcd of $(8, 10) = 2$ and $2 / 2$.
then $\frac{2}{8} \in Z_{10}$.

$\Rightarrow 2 = 8x$ $\qquad \frac{2}{8} = x$. then $x = 4 \in Z_{10}$

$2 = 32$.

Ques is $\frac{1}{5} \in Z_{11}$?

gcd of $(5, 11) = 1$ and $1/1$
so $\frac{1}{5} \in Z_{11}$.

Ques $\frac{2}{3} \in Z_{100}$?

gcd of $(3, 100) = 1$ & $1/2$
so $\frac{2}{3} \in Z_{100}$.

integral domain: if $a \cdot b = 0$ $\Rightarrow$ either $a = 0$ or $b = 0$ or both 0.

One-One mapping :- A mapping $T : A \to B$ is said to be one-one mapping if $T(x) = T(y)$ then $x = y$.

Ques. $T : Z \to Z$ defined by $T(x) = 2x$, $2 \in Z$ is one-one mapping?

Sol^n :- $T : Z \to Z$ defined by $T(x) = 2x$

let $T(x) = T(y)$

$\Rightarrow 2(x) = 2(y)$

$2(x-y) = 0$

$2 \neq 0$ then $x - y = 0$ [$\because (Z, +, \cdot)$ is an integral domain and $2(x-y) = 0$]

$x = y$

then T is one-one.

Ques. $T : Z_{10} \to Z_{10}$ defined by $T(x) = 2x$ is mapping is one-one?

Sol^n.

$T(x) = T(y)$

$2x = 2y$

$2(x-y) = 0$

$2 \neq 0 \Rightarrow$ ~~$x-y=0$~~

For e.g. $\to Z_{10} = \{0, 1, 2, \ldots 9\}$ if we choose any two no. like 4, 5 then $4 \cdot 5 = 20 = 0$ under Z_{10} so product of two non-zero no. is zero so it is not an integral domain

$(Z_{10}, +, \cdot)$ is not an integral domain.

$T(x) = 2x$

$0 \in Z_{10}$ then $T(0) = 2 \cdot 0 = 0$

$5 \in Z_{10}$ then $T(5) = 2 \cdot 5 = 10 \pmod{10} = 0$

so $T(0) = T(5)$

but $0 \neq 5$

so $T : Z_{10} \to Z_{10}$ is not one-one.

Z_6 is not ... because $2, 3 \in Z_6$ s.t. $2 \cdot 3 = 0$ where 2, 3 are non-zero.

Ques. $T : Z_6 \to Z_6$ defined by $T(x) = 5x$ is one - one mapping?

Sol^n:- let $T(x) = T(y)$
$5x = 5y$

Now $\frac{1}{5} \in Z_6$ [$\because$ gcd of $(5, 6) = 1$ & $1/1$].

then $\frac{1}{5} \cdot 5x = \frac{1}{5} \cdot 5y$

As value of $\frac{1}{5}$ in mod 6 is 5, i.e $\left[\because 5 \mid 1 \Rightarrow 1 = 5x,\ x = 5 \text{ under mod 6}\right]$

$5 \cdot 5x = 5 \cdot 5y$
$25x = 25y$
$1 \cdot x = 1 \cdot y$ $\quad [25 \pmod 6 = 1]$
$\Rightarrow x = y$
$\Rightarrow T$ is one - one.

Ques:- $T : Z_8 \to Z_8$ defined by $T(x) = 3x$ is one - one?

Sol^n:- let $T(x) = T(y)$
$3x = 3y$
Now $\frac{1}{3} \in Z_8$ then $\frac{1}{3} \cdot 3x = \frac{1}{3} \cdot 3y$ $\quad [\because \frac{1}{3} = 9]$ under mod 8.
$\Rightarrow 9x = 9y$
$\Rightarrow x = y$ then T is one - one.

Ques $T : Z_{20} \to Z_{20}$ defined by $T(x) = 6x$ is one - one?

Sol^n:- $0 \in Z_{20}$ then $T(0) = 6 \cdot 0 = 0$
$10 \in Z_{20}$ then $T(10) = 6 \cdot 10 = 60 = 0$
$T(0) = T(10)$ but $0 \neq 10$
so T is not one - one.

3 Aug, 2019

Equations

ODE PDE Int. Eqn COV

PART-B

ODE, PDE: 4-5 Ques ≃ 12-15 Marks

Int. Eqn, COV: ⇒ 3 Ques ≃ 9 Marks

PART-C

5 Ques

5 × 4.75 = 23.75

Int. Eqn: 4.75

COV: 4.75

50-55 Marks including PART-B & C

Book :- ODE: Boyce, Ross

PDE :- Denmeyer, Snedden

COV :- Hans Segan

ODE

I.) First order & Ist Degree Equation

$$\frac{dy}{dx} = f(x,y)$$

- Variable Separation
- Linear Diff. Eq^n
- Exact Diff. Eq^n

II.) First Order & Higher Degree

- Clairatt's Eq^n
- Singular Sol^n

III) Existence & Uniqueness Th^m (EUT)

– To solve Initial Value Problem (IVP), Ist Order

IV) Wronskian

V) Higher Order Diff. eq^n with constt. coefficient

VI) Second Order Linear Diff. eq^n with Variable Coeff.

- When one sol^n of Homogeneous eq^n is known
- Variation of Parameters (VOP) Method → (Green's Function
- Normal Form Method.

VII) Sturm Liouville Boundary Value Problem (BVP)

VIII) Green's functions

IX) System of Equation

— Solution using Eigenvalues.

XXX

DIFFERENTIAL EQUATIONS (ORDINARY) :-

$y \to$ dependent Variable

$x \to$ independent Variable

$$y = f(x)$$

$$\frac{dy}{dx}, \frac{d^2y}{dx^2}, \dots$$

$$F(x, y, y', y'', \dots) = 0$$

is diff. eqn.

→ When we have only one independent variable then it is ordinary Diff. eqn.

Def :- An equation which contains dependent Variable and its derivatives (w.r to indep. variables) is called Diff eqn.

for e.g i) $\frac{d^2y}{dx^2} - x\frac{dy}{dx} = e^x$ is ODE

ii) $\frac{d^3y}{dx^3} - y\frac{d^2y}{dx^2} + x^2\frac{dy}{dx} - 2y = \sin x$

or $F(x, y, y', y'', y''') = 0$

→ general form

Order of Eqn:- Highest Order Derivative.

Example:- 1) $\left(\frac{d^2y}{dx^2}\right)^1 - x\left(\frac{dy}{dx}\right)^1 = e^x$

Linear term

Order $= 2$

Linear term

When eqn has linear terms then it it Linear ODE.

2) $\left(\frac{d^3y}{dx^3}\right)^1 - (y)^1\left(\left(\frac{d^2y}{dx^2}\right)\right)^1 + x^2\left(\frac{dy}{dx}\right)^1 - 2(y)^1 = \sin x$

Linear term

Non-Linear

Linear

Linear

Order of Eqn $= 3$

This is Non-Linear term.

Linear Diff: Eqn:-

A diff. eqn in which dep. variable & its derivative occur only in degree 1.

NOTE :- Linearity & Non-Linearity are only defined by dependent Variable not by independent Variable.

E.g :- 1) $\left(\frac{d^3y}{dx^3}\right)^1 + x^2\left(\frac{d^2y}{dx^2}\right)^1 - (y)^2 = x$

$\left(\frac{d^3y}{dx^3}\right)$ → Linear, $\left(\frac{d^2y}{dx^2}\right)$ → Linear, $(y)^2$ → Non-Linear

So eqn is Non-Linear & its order is 3.

NOTE :- If an eqn contains transcendental function of dep. variable then such eqn is non-Linear

E.g :- $\sin y$, $\cos y$, e^y, $\log y$ → Transcendental functions

[∵ Their expressions contains non-Linear term

Ist ORDER DIFF. EQUATION

$$\frac{dy}{dx} = F(x, y) \quad \text{— (1)}$$

General form of any Ist Order Diff. eqn (Including Linear & Non-Linear)

E.g :- i) $\frac{dy}{dx} = xy^2$ is non linear.

ODE of Ist Order & Ist Degree :

General Form of such equation is

$$\frac{dy}{dx} = f(x,y) \quad - (1)$$

We want to find out a continuous function $y = \phi(x)$ which satisfies eqn (1), such function is called soln of eqn (1)

General Soln of eqn. (1) contains one arbitrary function i.e the general soln of (1) is given by $\boxed{\phi(x,y,c) = 0}$ — (2)

or $\boxed{\phi(x,y) = c}$

c - arbitrary constant

⇒ General soln is one which contains all possible solutions of eqn (1)

⇒ If we give particular value to the arbitrary constant in eqn (2), then we get particular solution.

NOTE :- In some diff eqns, we get some solutions which can not be obtained from general solution by giving some particular value to the arbitrary constant and which donot contain any arbitrary constant, such soln are called singular solution

* $\frac{dy}{dx} = f(x,y)$ — ① → [General Form]

We want to find out. $\phi(x,y,c) = 0$ — ②

[General solⁿ]

Eqⁿ ① can always be written as :-

$$M(x,y)\,dx + N(x,y)\,dy = 0 \quad \text{— ③}$$ Standard Format of ①

<u>We have following Methods to solve eqⁿ ① or eqⁿ ③</u>

- Variable Separable
- Homogeneous Equation
- Eqⁿ that can be made homogeneous
- Linear Diff. eqⁿ
- Bernouli Equation
- Exact Diff. eqⁿ.

I) <u>Variable Separable</u> :-

If we can write given eqⁿ as

$$\phi_1(x)\,dx + \phi_2(y)\,dy = 0 \quad \text{— ①}$$

then we say that our variables are separated

Now we integrate ①, to find general solⁿ.

E.g.:- $\frac{dy}{dx} = \frac{y}{x}$ — ①

$\frac{1}{y}\,dy = \frac{1}{x}\,dx$ — ② → Variables are separated

Integrate ② :-

$\int \frac{1}{y}\,dy = \int \frac{1}{x}\,dx + \log c$

where c is arbitrary constant

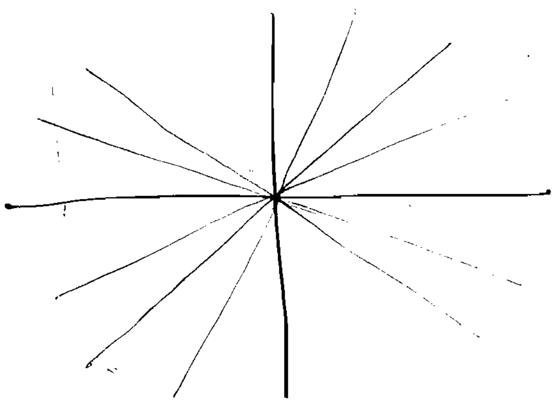
$\log y = \log x + \log c$

$\boxed{y = cx}$ → General soln

$c = 1$, $y = x$ → Particular soln

$c = 2$, $y = 2x$ → Particular soln

$c = 3$, $y = 3x$ → Particular soln



family of curves (straight lines) as soln of eqn ①

→ These solutions are called solution curves.

NOTE:- i) Solution curves (integral curves) are same because we get soln by integration.

ii) In ODE, these called solution curves

But in PDE, we get solution surfaces.

26 July, 2017

Syllabus :-

1. Point set topology on $\mathbb{R}$.
2. Countability of sets
3. Sequence & series of reals.
4. Functions
5. Limit continuity, Uniform continuity
6. Differentiablity
7. Riemann Integration
8. Improper Integral
9. Function of bounded variation
10. Sequence & series of functions (Uniform convergence)
11. Several variable calculus
12. Measure Theory

Book :- S.C. Malik & Savita Arora (Mathematical Analysis)

or R.G. Bartle

Desmos graphing

Some Standard Notations :- (Reserved | fixed)

$\mathbb{N}$	Set of Natural numbers
$\mathbb{N}_0$	Set of whole numbers
$\mathbb{Z}$	Set of integers
$\mathbb{Q}$	Set of rational numbers
$\mathbb{Q}^c$	Set of irrational numbers
$\mathbb{R}$	Set of Real Numbers
$\mathbb{C}$	Set of complex numbers
$+\infty$	Arbitrary large representation
$-\infty$	Arbitrary small representation
$\bar{A}$	closure of A
A°	Interior of A
∂A	Boundary of A
A'	Derived set
Iso(A)	Isolated points of A
s.t.	such that
s.t.b.	said to be
bdd.	Bounded
unbdd	Unbounded
n.b.d.	neighbourhood
seq.	sequence
cgt	convergent
dgt	divergent
$\overline{\lim}$	limit superior
$\underline{\lim}$	limit inferior
sup(A) | l.u.b.	Supremum
inf(A) | g.l.b.	infimum
max A	maximum
min A	minimum

$\nexists$	doesn't exist
$\exists!$	exist uniquely
$\forall$	for all
$\divideontimes$	contradiction
ϵ	epsilon
$\in$	belongs to
$\notin$	doesn't belong
$\cup$	Union
$\cap$	Intersection
$\bigcup\limits_{\alpha\in\Delta}$	Arbitrary Union
$\bigcap\limits_{\alpha\in\Delta}$	Arbitrary intersection
$\bigcup\limits_{n=1}^{\infty} A_n$	~~Arbitrary~~ countable Union
$\bigcap\limits_{n=1}^{\infty} A_n$	countable intersection
$\bigcup\limits_{\cancel{n}=1}^{n} A_n$	finite Union
$\bigcap\limits_{K=1}^{n} A_n$	finite intersection

$S = P \Rightarrow q$ $\quad$ $\overset{\text{sufficient}}{P}$ implies $\overset{\text{necessary}}{q}$

converse of $S = q \Rightarrow P$ (converse may not be true)
(prove by counter eg.)

$\sim q \Rightarrow \sim P$ $\quad$ contrapositive is always true

P and q $\quad$ if ~~p~~ both P & q. both holds

$P \underline{\underline{\text{or}}} q$ $\quad$ P or q or both

$P \Leftrightarrow q$ $\quad$ iff (if and only if
(Necessary and sufficient).

A.M. $(a_1, \ldots a_n)$ $a_1 + a_2 + \cdots + a_n / n$ (Arithmetic Me

G.M. $(a_1 \ldots a_n)$ $(a_1 \cdot a_2 \cdots a_n)^{1/n}$ (Geometric Mean)

H.M. $(a_1, \ldots a_n)$ $\cancel{n/a_1}$ $\dfrac{n}{\frac{1}{} + \frac{1}{} + \cdots \frac{1}{n}}$ (Harmonic Mean)

$S.M.(a_1 \ldots a_n)\ \sqrt{a_1^2 + a_2^2 + \cdots + a_n^2 / n}$ (Square Mean)

where $a_1, a_2, \ldots a_n \in \mathbb{R}^+$

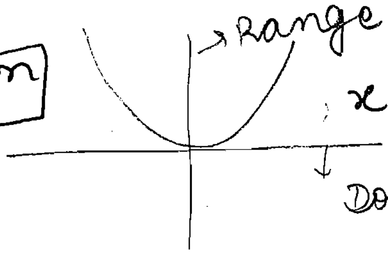
$\operatorname{Min}\{a_1, a_2, \ldots, a_n\} \le H.M.(a_1, a_2, \ldots a_n) \le G.M.$

$\le A.M.(a_1, \ldots a_n) \le S.M.(a_1, \ldots, a_n)$

$\le \operatorname{Max}\{a_1, a_2, \ldots a_n\}$

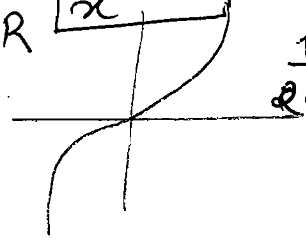
even root → Non-ve Real no.

$\boxed{x^{2n}}$



$x^{\frac{1}{2n}} : [0, \infty] \to \mathbb{R}$

$\boxed{x^{2n+1}}$

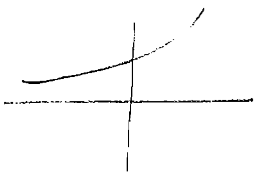


$\frac{1}{x^{2n+1}} : \mathbb{R} \to \mathbb{R}$

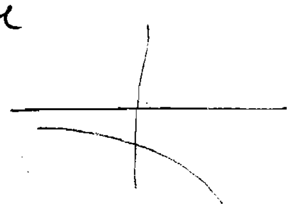
$f(x) \to -f(x)$ Reflection about x-axis

$f(x) \to f(-x)$ Reflection about y-axis

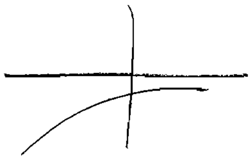
e^x



e^{-x}



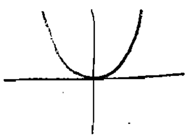
$-e^{-x}$



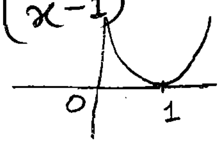
$f(x) \to f(x+a)$ Shift left or right

$\downarrow 0 \qquad \downarrow -a$ (Shift to $-a$)

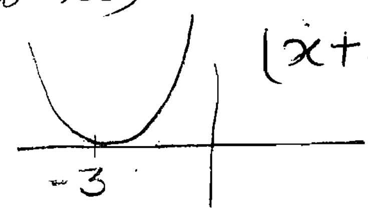
x^2



$(x-1)^2$

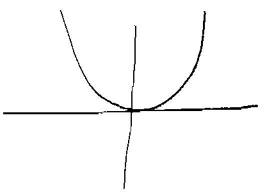


$(x+3)^2$

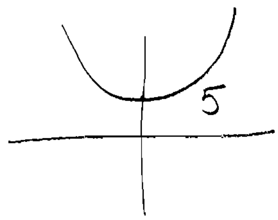


$f(x) \to f(x) + a$ Shift up or down

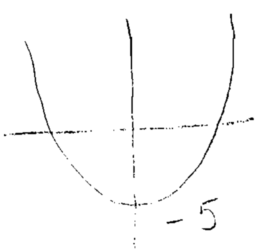
x^2

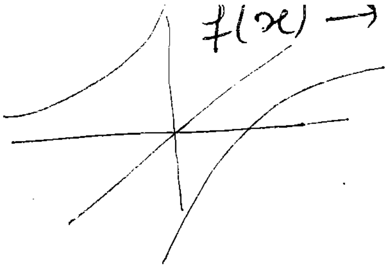


$x^2 + 5$



$x^2 - 5$





$f(x) \to f^{-1}(x)$ Reflection about line $y = x$

$\downarrow$

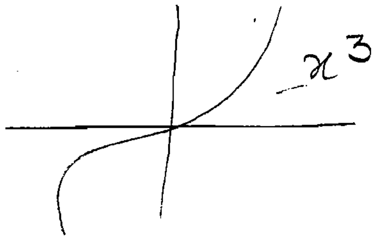
inverse

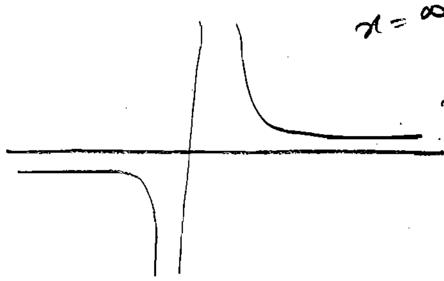
$f(x) \to \frac{1}{f(x)} \to$ reverse

$x = 0$ then $\frac{1}{x} = \infty$

$x = \infty$ then $\frac{1}{x} = 0$

$\frac{1}{x^3}$





$\frac{1}{x^3}$

Even functions are symmetric about y-axis.

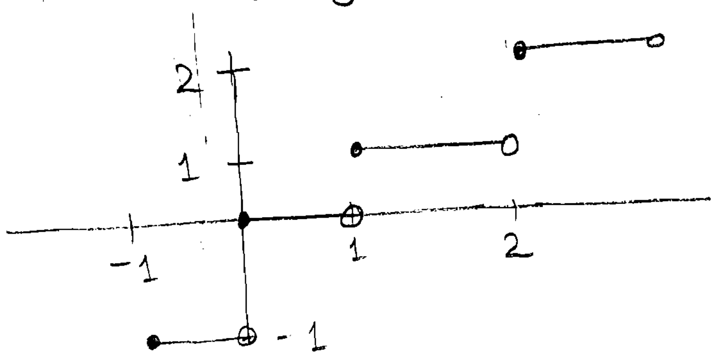
Odd functions are symmetric about opposite ~~quarc~~ quadrant.

$[x]$ greatest integer

$\{x\}$ fractional part of x.

Rel^n. b/w greatest integer & fractional

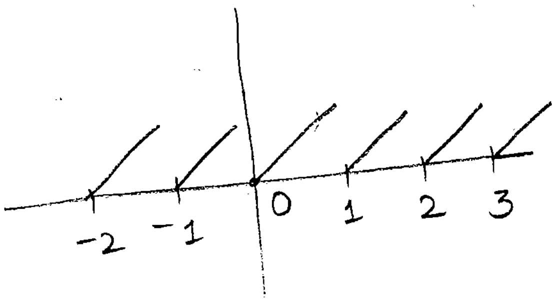
$x - \{x\} = [x]$ or $x - [x] = \{x\}$



Graph of greatest integer

$$[x] = \begin{cases} 0 & 0 \le x < 1 \\ 1 & 1 \le x < 2 \\ 2 & 2 \le x < 3 \\ \vdots & \\ -1 & -1 \le x < 0 \end{cases}$$

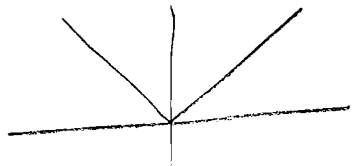
$$\{x\} = x - [x] = \begin{cases} x & 0 \le x < 1 \\ x-1 & 1 \le x < 2 \\ x-2 & 2 \le x < 3 \\ \vdots & \\ x+1 & -1 \le x < 0 \\ \vdots & \end{cases}$$



Periodic function
with 1 as period
(Value lies b/w 0 & 1)

$|x|$ = Modulus of x (Absolute value of x)

$$= \begin{cases} x & x \geq 0 \\ -x & -x \le 0 \end{cases}$$



Vacuously True :- if having no counter example.

eg. Every four legs person is pakistani.

Set :- A well defined collection of distinct objects. clear cut or defined in actual sense.

$\{x_1, x_2, \ldots \mid P(x_1), P(x_2), \ldots\}$

collection	Set
ϕ	✓
$\{\phi\}$	✓
$\mathbb{N}$	✓
$\mathbb{N}_0$	✓
$\mathbb{Z}$	✓
$\mathbb{Q}$	✓
$\mathbb{Q}^c$	✓
$\mathbb{R}$	✓
$\mathbb{C}$	✓
$\{x \in \mathbb{R} : x^2 \geq 0\} = \mathbb{R}$	✓
$\{x \in \mathbb{R} : x^2 > 0$	×
$\{1, 2, 3\}$	✓
$\{1, 2, 3, 4$	×
$\{x \in \mathbb{R} : x > 0 \,\&\, x < 0\}$	✓

$|\phi| = 0$

$|\{\phi\}| = 1$

every Real No.

collection of fans in this class room ✓
collection of A.C. in this class room ✓
collection of boys students in class room ✓
collection of girls student in class room ✓
collection of intelligent students in this class room ×
collection of smart boys in class ×
collection of beautiful boys in class ×
collection of " girls in class ×
" " M.sc. degree holder " " ✓
" " ~~fo~~ Phd. degree holder ✓

Set bounded above :- A set $A \subseteq \mathbb{R}$ is said to be bounded above if $\exists$ $K \in \mathbb{R}$ s.t. $x \leq K \;\; \forall \; x \in A$.
otherwise set is said to be unbounded above. $K \rightarrow$ an upper bound of A.

$\boxed{\text{Note}}$:- (i) $K' > K$ is an upper bound of A.
(ii) bdd above $\Leftrightarrow$ infinite no. of upper bounds.
(iii) Not bdd above $\Leftrightarrow$ No upper bounds
(iv) bdd above $\Rightarrow$ No largest upper bound.
★★(v) Every non-empty bdd above set has l.u.b. in $\mathbb{R}$. (completeness property of $\mathbb{R}$)
<u>or</u> Real line has no gap.

bdd above + जहां set के elt. खत्म होते हैं वहां Upper Bound.

Set	bdd. above	Upper bounds
1. ϕ 5, 7 –	✓	$\mathbb{R}$
2. $\{a\}$	✓	$[a, \infty)$
3. $\{a_1, a_2, \cdots a_n\}$	✓	$[a_n, \infty)$
4. $\mathbb{N}$	×	–
5. $\mathbb{N}_0$	×	–
6. $\mathbb{Z}$	×	–
7. $\mathbb{Q}$	×	–
8. $\mathbb{Q}^c$	×	–
9. $\mathbb{R}$	×	–
10. $(0, 1)$	✓	$[1, \infty)$
11. $[0, 1]$	✓	$[1, \infty)$
12. $(0, \infty)$	×	–
13. $(-\infty, 0)$	$\cancel{\times}$ ✓	$[0, \infty)$
14. $(-\infty, 0]$	$\cancel{\times}$ ✓	$[0, \infty)$
15. $(0, 1) \cap \mathbb{Q}$	✓	$[1, \infty)$
16. $(0, 1) \cap \mathbb{Q}^c$	✓	$[1, \infty)$

Complex Analysis

marks : 30

Syllabus :

UNIT-01

Complex Numbers

UNIT-02

Analytic function

UNIT-03

Complex Integration.

* UNIT-04

Important theorem and results.

UNIT-05

Bilinear (Mobius) Transformations

Book- foundation of complex Analysis (by Ponnusar

* Some Standard Notations :

- $\mathbb{N} \rightarrow$ Set of natural numbers
- $\mathbb{N}_0 \rightarrow$ Set of Whole numbers
- $\mathbb{Z} \rightarrow$ Set of integers.
- $\mathbb{Q} \rightarrow$ set of rational numbers
- $\mathbb{Q}^c \rightarrow$ set of irrational numbers
- $\mathbb{R} \rightarrow$ Set of real numbers
- $\mathbb{C} \rightarrow$ set of complex numbers. (complex plane/finite complex plane)
- $\mathbb{R}_\infty = \mathbb{R} \cup \{+\infty, -\infty\}$ (extended real line)
- $\mathbb{C}_\infty$ – Extended complex plane
  $\mathbb{C} \cup \{\infty\}$
- $z = x + iy \longrightarrow$ complex number
- $Re(z) \rightarrow$ real part of z
- $Im(z) \rightarrow$ Imaginary part of z

- $\bar{z}$ — conjugate of z
- $i = \sqrt{-1}$ – iota
- $C \longrightarrow$ curve
- $\int_C f(z)\,dz \rightarrow$ integration of f over C.

- $\oint_C f(z)\,dz \rightarrow$ integration of f over closed curve C.

- $H(D) \rightarrow$ Set of Holomorphic fu^n (analytic/regular) on D.
- $D \longrightarrow$ Domain

$\underline{10^{th}}$

$$a = b \neq 0$$
$$a^2 = ab$$
$$a^2 - b^2 = ab - b^2$$
$$\cancel{(a-b)}(a+b) = b\cancel{(a-b)}$$
$$a+b = b \quad \text{(wrong step)}$$
$$2b = b$$
$$2 = 1$$

$\times$

$\underline{12^{th}}$

$$-1 = i^2 = i \cdot i$$
$$= \sqrt{-1} \cdot \sqrt{-1}$$
$$= \sqrt{(-1)(-1)} \quad \text{(wrong step by prop of } f(z)=\sqrt{z}\text{)}$$
$$= \sqrt{1}$$
$$= 1$$

$\times$

- **Addition :-**

$$z_1 + z_2 = (x+iy) + (a+ib)$$
$$= (x+a) + i(y+b)$$

- **Additive identity:**

$$z = 0 = 0 + i0$$

- **Additive Inverse:** $-z = -(x+iy) = -x - iy$

- **Multiplication:** $(a+ib)(c+id) = (ac-bd) + i(ad+bc)$

- **Multiplicative identity:** $1 = 1 + i0$

- **Multiplicative Inverse:** $0 \neq z = a+ib$

$$z^{-1} = \frac{1}{a+ib} \times \frac{a-ib}{a-ib} = \frac{a}{a^2+b^2} - i\frac{b}{a^2+b^2}$$

* **Ordered field:**

A field $\mathbb{F}$ is said to be an ordered field if $\exists$ a non-empty subset P of $\mathbb{F}$ satisfying the following

1) $0 \notin P$
2) closed under addition: $x, y \in P \Rightarrow x+y \in P$.
3) closed under Multiplication: $x, y \in P \Rightarrow x \cdot y \in P$
4) for any $x \in \mathbb{F}$, exactly one of the following holds
$x = 0$ or $x \in P$ or $-x \in P$.

* $(\mathbb{R}, +, \cdot)$ is an ordered field.
$P = \mathbb{R}^+$
* $(\mathbb{Q}, +, \cdot)$ is an ordered field.
$P = \mathbb{Q}^+$
* $(\mathbb{Q}(\sqrt{2}), +, \cdot)$ is an order field.

* $(\mathbb{C}, +, \cdot)$ is not an order field.

$i \neq 0$, $\quad i \in P \qquad -i \in P$.

$i^2, i^4 \in P \qquad -i^2, (-i)^4 \in P$

$-1, 1 \in P \qquad -1, 1 \in P$.

(contradics prop. iv) $\quad \times \qquad \times$ (contradiction)

* $a+ib \nleq c+id$ $\quad |a+ib| \leq |c+id|$

$\not\geq \qquad \Leftrightarrow \quad a = c \qquad \geq$

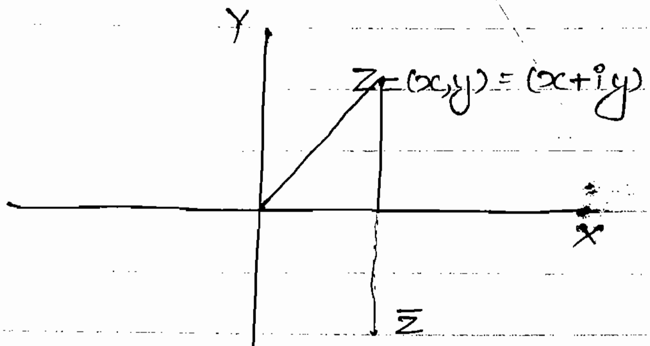
$\not> \qquad\qquad b = d \qquad <$

$\not< \qquad\qquad\qquad \geqq$

$\because$ (complex is not orderded fied) $\quad$ (Real is an ordered fied)

* Conjugate of Complex Number:

$$\bar{z} = \overline{x+iy} = \text{conjugate of } z$$
$$= x - iy$$



Reflection about x-axis.

* Properties:

1) $\overline{z_1+z_2} = \bar{z}_1 + \bar{z}_2$

2) $\overline{\sum_{i=1}^{n} z_i} = \sum_{i=1}^{n} \bar{z}_i$

3) $\overline{z_1 \cdot z_2} = \bar{z}_1 \cdot \bar{z}_2$

4) $\overline{\prod_{i=1}^{n} z_i} = \prod_{i=1}^{n} \bar{z}_i$

5) $\bar{\bar{z}} = z$

6) $\overline{\left(\frac{z_1}{z_2}\right)} = \frac{\bar{z}_1}{\bar{z}_2}$

7) $z\bar{z} = |z|^2$

8) $\frac{z+\bar{z}}{2} = \mathrm{Re}(z)$

9) $\frac{z-\bar{z}}{2i} = \mathrm{Im}(z)$

10) $\bar{z} = z \iff z \in \mathbb{R}$
$\bar{z} = -z \iff z \in i\mathbb{R}$.

Q: $P(z) \in \mathbb{R}[x]$

$P(z) = 0 \Rightarrow P(\bar{z}) = 0$

$\rightarrow$ $P(z) = 0 = a_0 + a_1 z + a_2 z^2 + \cdots + a_n z^n \quad ; \; a_i \in \mathbb{R}$

$$P(\bar{z}) = \bar{0} = \overline{a_0 + a_1 z + \cdots + a_n z^n}$$
$$0 = a_0 + a_1 \bar{z} + \cdots + a_n \bar{z}^n$$
$$= p(\bar{z})$$

$P(z) \in \mathbb{C}[x]$; $P(z) \notin \mathbb{R}[x]$.
$\exists\ z \in \mathbb{C}$ s.t $P(z) = 0$ but $P(\bar{z}) \neq 0$

$$p(z) = (z-z_1)(z-\bar{z}_1)(z-z_2)(z-\bar{z}_2)\ldots(z-z_k)(z-\bar{z}_k)$$
$$= (z^2-(z_1+\bar{z}_1)z + z_1\bar{z}_1)(z^2-(z_2+\bar{z}_2)z + z_2\bar{z}_2)\ldots(z^2-(z_k+\bar{z}_k)z + z_k\bar{z}_k)$$

↓ Real ↓ Real ↓ Real

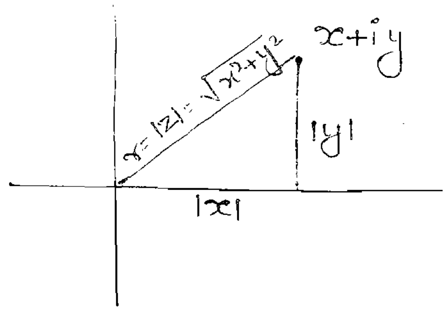
But $p(z) \notin \mathbb{R}[x]$.

* Modulus (Absolute value):

$Z = x+iy$ (cartesian form)

(modulus of z)

$$|z| = \sqrt{x^2+y^2}$$
$$= r$$



Properties:

i) $|z| \geq 0$, $|z| = 0 \iff z = 0$

ii) $|z| = |\bar{z}|$

iii) $z\bar{z} = |z|^2$

iv) $|z_1 z_2| = |z_1||z_2|$

v) $|z_1 - z_2|$ = distance between z_1 and z_2.

vi) $Re(z) \leq |Re(z)| \leq |z|$
$\quad x \qquad |x| \qquad \sqrt{x^2+y^2}$

vii) $Im(z) \leq |Im(z)| \leq |z|$

viii) $Re(z) + Im(z) \leq |Re(z)| + |Im(z)| \leq 2|z|$

Brown & Churchill ix) $|Re(z)| + |Im(z)| \leq \sqrt{2}|z|$
$\quad |x| + |y| \leq \sqrt{2}\sqrt{x^2+y^2}$

$$|x| + |y| \leq \sqrt{2}\sqrt{x^2+y^2}$$

$$\Rightarrow |x|^2 + |y|^2 + 2|x||y| \leq 2(|x|^2 + |y|^2)$$

$$\Rightarrow |x|^2 + |y|^2 - 2|x||y| \geq 0$$

$$\Rightarrow (|x| - |y|)^2 \geq 0.$$

for max value

(X) $|z_1 + z_2| \leq |z_1| + |z_2|$ (Triangular inequality)

$$|z_1+z_2|^2 = (z_1+z_2)(\overline{z_1+z_2}) \qquad \{\because z\bar{z} = |z|^2$$
$$= (z_1+z_2)(\bar{z}_1 + \bar{z}_2)$$
$$= |z_1|^2 + |z_2|^2 + z_1\bar{z}_2 + z_2\bar{z}_1$$
$$= |z_1|^2 + |z_2|^2 + z_1\bar{z}_2 + \overline{z_1\bar{z}_2}$$
$$= |z_1|^2 + |z_2|^2 + 2\,\mathrm{Re}(z_1\bar{z}_2)$$
$$\leq |z_1|^2 + |z_2|^2 + 2|z_1\bar{z}_2| \qquad \{\because \mathrm{Re}(z) \leq |z|$$
$$= (|z_1| + |z_2|)^2.$$

$$\Rightarrow |z_1+z_2| \leq |z_1| + |z_2| \qquad \left\{\begin{array}{l}\because x \leq y \\ \Rightarrow \sqrt{x} \leq \sqrt{y}\end{array}\right.$$

(XI) $|z_1 - z_2| \leq |z_1| + |z_2|$.

$$\because (|z_1 + (-z_2)| \leq |z_1| + |-z_2|$$
$$= |z_1| + |z_2|$$

for min. value

(XII) $\big||z_1| - |z_2|\big| \leq |z_1 - z_2|$

? $\big||z_1| - |z_2|\big| \leq |z_1 + z_2|$

proof:

$$|z_1| = |z_1 - z_2 + z_2| \leq |z_1 - z_2| + |z_2|$$

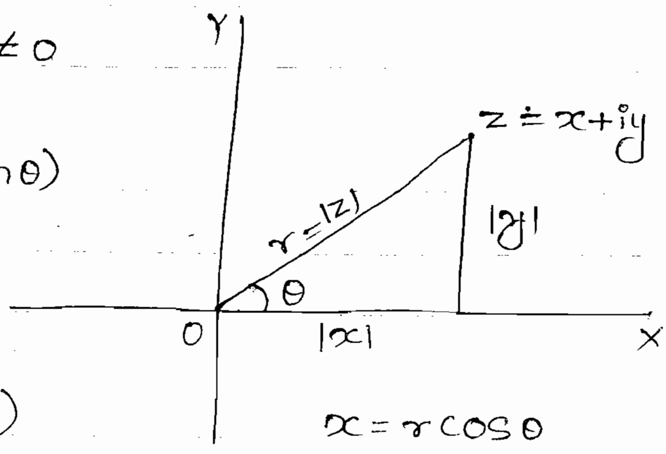
$$\Rightarrow |z_1| - |z_2| \leq |z_1 - z_2| \quad \text{—— (i)}$$

$$\Rightarrow |z_2| - |z_1| \leq |z_1 - z_2| \quad \text{—— (ii)}$$

$$\therefore \big||z_1| - |z_2|\big| \leq |z_1 - z_2|$$

* Polar form of Complex Number :-

$$z = x + iy \neq 0$$



$$z = r(\cos\theta + i\sin\theta)$$
$$z = re^{i\theta}$$

(Polar form of complex number)

$$x = r\cos\theta$$
$$y = r\sin\theta$$

r = modulus of z

θ = argument of z ($\arg . z$) (general argument).

$$z = |z| e^{i\theta}$$
$$= |z| e^{i(\theta + 2n\pi)} \quad ; n \in \mathbb{Z}$$

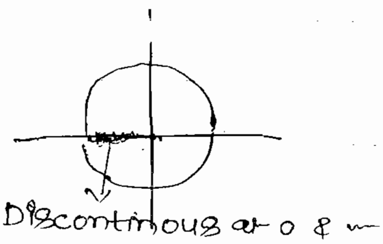
$$\arg z = \{\theta + 2n\pi, n \in \mathbb{Z}\}$$
= infinite set.

$f(z) = \arg . z$ is an infinite valued function.

$f(z) = \arg . z = \theta$, $\theta \in (k, k+2\pi]$. is single valued function.

$f(z) = \operatorname{Arg} z = \theta_1$; $\theta_1 \in (-\pi, \pi]$

? = Principal argument of z



$\arg(z)$	$\operatorname{Arg}(z)$
* general argument	* Principal Argument
* Infinite valued function	* Single valued function